DIFFERENTIAL EQUATIONS

Lecture 1 Summary

- A differential equation is simply an equation which consists of one or more derivatives.
- We learnt how to solve 3 special types of differential equations
  \[ \frac{dy}{dx} = f(x) \implies \text{Integrate both sides w.r.t. } x \Rightarrow \int \frac{dy}{dx} \, dx = \int f(x) \, dx \implies y = F(x) + C \]
  \[ \frac{d^2y}{dx^2} = f(x) \implies \text{Integrate both sides twice w.r.t. } x \Rightarrow \frac{dy}{dx} = F(x) + C \implies y = H(x) + Cx + D \]
  \[ \frac{dy}{dx} = g(y) \implies \text{Flip then integrate w.r.t. } y \Rightarrow \frac{dx}{dy} = \frac{1}{g(y)} \implies x = K(y) + C \]
- General solution is an answer containing one or more arbitrary constants.
- Particular solution is an answer containing no arbitrary constants.
- Must learn Example 7 well.

Lecture 2 Summary

- Substitution method is very similar to what we did in integration substitution.
  - Note that after substitution, the new differential equation will likely be very simply – in one of the three forms we learnt.
  - After solving the new differential equation, we must remember to change the variables back to the old variables.
- A differential equation solution can be viewed as a solution curve.
- When we have a solution curve given by \( y = -x - \frac{x^2}{2} + C \), the curves that we sketch are known as family of solution curves (Example 11).
On labeling, it is sufficient to just label each family of solution curves with the value of C used.

A method that check whether one family of solution curve lies above or below another family of solution curve is to take the difference:

Let curve 1 be: \( y = -x - \frac{x^2}{2} + C_1 \)

Let curve 2 be: \( y = -x - \frac{x^2}{2} + C_2 \)

If \( C_2 > C_1 \), then the difference (curve 2 – curve 1) =

\[
\left( -x - \frac{x^2}{2} + C_2 \right) - \left( -x - \frac{x^2}{2} + C_1 \right) = C_2 - C_1
\]

Then we know straightaway that curve 2 is above curve 1 for all \( x \).
Lecture 3 Summary

- Basic facts that need to know
  o When $y$ is proportional to $x$, it means $y = kx$.
  o When $y$ is inversely proportional to $x$, it means $y = \frac{k}{x}$.
  o The rate of $y$ is denoted by $\frac{dy}{dt}$, and rate always means we are referring to time.
  o At certain instant, when $y$ is increasing at a rate of 0.01, we have $\frac{dy}{dt} = 0.01$.
    But when decreasing at a rate of 0.01, we have $\frac{dy}{dt} = -0.01$ (there is a negative sign to denote decreasing).

- Important system (refer to Example 14):
  Net rate of volume = (Rate of volume IN) – (Rate of Volume OUT).
  For example, if water flows into the tank via pipe 1 at a constant rate of $a$, and flows into the tank at a constant rate of $b$ via pipe 2. Water also flows out via a single pipe at a constant rate of $c$, and the tank is so old that there is a leak of water at a constant rate of $d$, then $\frac{dV}{dt} = (a+b)-(c+d)$.

- Example 14 solution details:
  After solving the equation $\frac{dx}{dt} = -k \left(2x-1\right)$ --- (2), we have $t = -\frac{1}{2k} \ln |2x-1| + C$. 
  At $t = 0, x = 0.75, 0 = -\frac{1}{2k} \ln |2(0.75) - 1| + C \Rightarrow C = \frac{1}{2k} \ln (0.5)$
  At $t = 0, x = 0.75, \frac{dx}{dt} = -0.01$, from (2), $-0.01 = -k \left(2(0.75)-1\right) \Rightarrow k = 0.02$
  Thus, $C = \frac{1}{2(0.02)} \ln (0.5) = 25 \ln (0.5)$
  Substitute found values of $k$ and $C$ into $t = -\frac{1}{2k} \ln |2x-1| + C$
  $\Rightarrow t = -25 \ln |2x-1| + 25 \ln (0.5)$
  $\Rightarrow \ln |2x-1| - \ln (0.5) = -\frac{t}{25}$
  $\Rightarrow \ln \left| \frac{2x-1}{0.5} \right| = -\frac{t}{25}$
  When $x = 0.55, \Rightarrow \ln \left| \frac{2(0.55)-1}{0.5} \right| = -\frac{t}{25} \Rightarrow t = 40.2$ s (to 3 s.f.)
Example 15 solution details:

We have solved the differential equation to obtain \( t = \frac{1}{k} \ln |p| + C \).

Because \( p \) represents the population, we have \( t = \frac{1}{k} \ln (p) + C \) (we rid the modulus).

In 1970 \( \Rightarrow t = 0, \ p = 2074500 \Rightarrow 0 = \frac{1}{k} \ln (2074500) + C \) ---- (1)

In 1990 \( \Rightarrow t = 20, \ p = 3074100 \Rightarrow 20 = \frac{1}{k} \ln (3074100) + C \) ---- (2)

(2) – (1): \( \Rightarrow 20 = \frac{1}{k} \left[ \ln (3074100) - \ln (2074500) \right] \)

\( \Rightarrow k = \frac{1}{20} \ln \left( \frac{10247}{6915} \right) = 0.0196646007 \) and substitute into say (1): \( C = -739.6657026 \).

With these two values found, we finally have the particular solution as \( t = \frac{1}{0.0196646007} \ln (p) - 739.6657026 \).

In 2008 means \( t = 38, \ \Rightarrow 38 = \frac{1}{0.0196646007} \ln (p) - 739.6657026 \)

\( \Rightarrow \ln (p) = 0.0196646007 (38 + 739.6657026) \)

\( \Rightarrow p = e^{0.0196646007(777.6657026)} = 4379676.932 \)

To 3 s.f., the answer for the population in Year 2008 is 4380000.